Relative Sensitivity: A Dynamic Closed-Loop Interaction Measure and Design Tool

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Recently a new concept, block relative gain (BRG), has been introduced as a synthesis tool for the systematic generation and selective screening of alternative decentralized control structures (Manousiouthakis et al., 1986). BRG generalizes Bristol's relative gain array (RGA; Bristol, 1966) to block pairing of inputs and outputs that are not necessarily single-input/single-output (SISO) pairings. However, strictly speaking, both RGA and BRG are steady state interaction measures. In a more recent work (Arkun, 1987) this shortcoming has been partially alleviated by introducing the new dynamic BRG (and the dynamic RGA for that matter). The dynamic BRG clearly identifies the stability margins of decentralized controllers and expresses how the individual plant subsystems respond to their own set points. However it does not give any useful information about the direction and magnitude of dynamic interactions among the subsystems. Therefore it cannot describe how a set point change in one subsystem affects the performance of other subsystems. More important, since it depends on the controller tuning it cannot easily be used to select control structure pairings. The objective of this paper is to fill these gaps by introducing the relative sensitivity as a new dynamic closed-loop interaction measure for analysis and design of decentralized control structures. In contrast with other dynamic interaction measures (e.g., Rijnsdorp's interaction measure, diagonal dominance, structured singular value interaction measure), the relative sensitivity is a closed-loop performance measure while others are based on or are derived from closed-loop stability considerations; as such, they cannot directly assess the true closed-loop performance.

The Relative Sensitivity Concept

Consider the decentralized internal model control (IMC) structure given in Figure 1. For convenience the Laplace variable s will be dropped unless otherwise noted. The plant transfer function matrix G is in block partition form:

$$\mathbf{G} = \left[\begin{array}{c|cc} G_{11} & G_{12} & G_{1k} \\ \hline G_{21} & G_{22} & \\ & & \cdot \\ G_{k1} & & G_{kk} \end{array}\right] \triangleq \left[\begin{array}{c|c} G_{11} & G_{cr_{11}} \\ \hline G_{cd_{11}} & G_{c_{11}} \end{array}\right] \qquad (1)$$

where $G_{c_{11}}$, $G_{cr_{11}}$, $G_{cd_{11}}$ are respectively the complement, the complement right, and the complement down of the first subsystem G_{11}

The diagonal blocks, G_{ii}'s (each of which is an $n_i \times n_i$ matrix), represent the subsystems that are under decentralized control. When each subsystem is treated in isolation (i.e., is disconnected from the rest of the plant) its set point response is given by

$$y_i = G_{ii} G_{Ii} r_i \triangleq H_i r_i \quad i = 1, 2, \ldots, k \qquad (2)$$

where the subsystem controller $G_{Ii} = G_{ii}^{-1} H_i$ is the usual IMC controller, and H_i is called the achievable performance of the ith subsystem in isolation. As shown in Figure 1, both the plant controller and the predictive model are decentralized.

Within this general framework we can now introduce the relative sensitivities.

Definition of relative sensitivities

In Figure 1 r_1 and y_1 are the set points and the outputs of the first subsystem G_{11}, respectively. Similarly, $r_{c_{11}} = [r_2, r_3, \ldots, r_k]^T$ and $y_{c_{11}} = [y_2 y_3 \ldots y_k]^T$ are the set points and the outputs of the complement of G_{11}. The set point response for the whole plant is given by the block partition form

$$\begin{bmatrix} y_1 \\ y_{c_{11}} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{1c_{11}} \\ Y_{c_{11}1} & Y_{c_{11}c_{11}} \end{bmatrix} \begin{bmatrix} r_1 \\ r_{c_{11}} \end{bmatrix} \qquad (3)$$

which is compatible with the partitioned plant, Eq. 1.

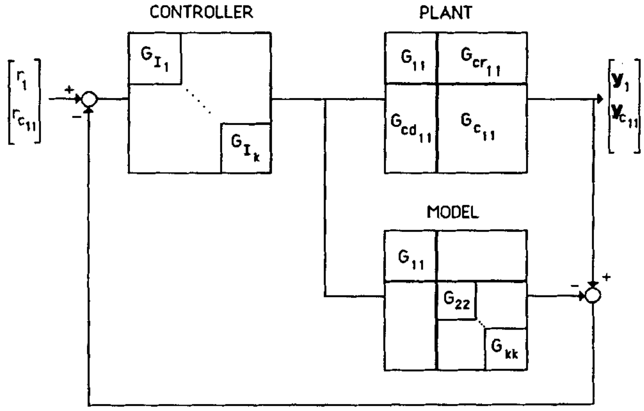

Figure 1. Decentralized IMC.

The relative sensitivity between the first subsystem G_{11} and its complement $G_{c_{11}}$ is defined by:

$$S_{c_{11}1} \triangleq Y_{c_{11}1}Y_{11}^{-1} = \begin{bmatrix} \left(\frac{\partial y_2}{\partial r_1}\right)\cdot\left(\frac{\partial y_1}{\partial r_1}\right)^{-1} \\ \left(\frac{\partial y_3}{\partial r_1}\right)\cdot\left(\frac{\partial y_1}{\partial r_1}\right)^{-1} \\ \vdots \\ \left(\frac{\partial y_k}{\partial r_1}\right)\cdot\left(\frac{\partial y_1}{\partial r_1}\right)^{-1} \end{bmatrix} \triangleq \begin{bmatrix} S_{21} \\ S_{31} \\ \vdots \\ S_{k1} \end{bmatrix} \quad (4)$$

where $S_{j1} \in C^{n_j \times n_1}$.

The other relative sensitivities $S_{c_{ii}i}$, $i = 2, 3, \ldots, k$, are similarly defined. All these relative sensitivities make up the columns of the relative sensitivity matrix denoted by S:

$$\mathbf{S} = \begin{bmatrix} I & S_{12} & & S_{1k} \\ S_{21} & I & & \vdots \\ S_{k1} & S_{k2} & \cdots & I \end{bmatrix} \quad (5)$$

where

$$S_{ji} = \left(\frac{\partial y_j}{\partial r_i}\right)\left(\frac{\partial y_i}{\partial r_i}\right)^{-1} \quad i, j = 1, 2, \ldots, k. \quad (6)$$

From Eq. 6 it is clear that S_{ji} is a dynamic (i.e., frequency-dependent) measure describing the effect of interactions on closed-loop performance. More specifically, S_{ji} expresses how much the jth subsystem is excited relative to the response of the ith subsystem when a set point change is made in the ith subsystem. For a totally noninteracting system $S = I$, and for a one-way interacting system S is either upper or lower triangular, depending on the direction of interactions.

Computation of relative sensitivities

The following is true for Figure 1:

$$Y = G \cdot bd[G_{Ii}]\{I + (G - bd[G_{ii}])\, bd[G_{Ii}]\}^{-1} \quad (7)$$

$$= G\{bd[H_i^{-1}G_{ii}] + (G - bd[G_{ii}])\}^{-1} \quad (8)$$

where bd stands for block diagonal. Applying the results given in Kailath (1980) for the inverse of block matrices to Eq. 8 and using Eqs. 3 and 4, one gets the relative sensitivity between G_{11} and its complement $G_{c_{11}}$:

$$S_{c_{11}1} = -bd(H_i - I)_{i\neq 1}\{I + [\mathrm{BRG}_{c_{11}}^{-1} - I)G_{c_{11}} + (G_{c_{11}} - G_{b_{11}})]bd(G_{ii}^{-1}H_i)_{i\neq 1}\}^{-1} G_{cd_{11}}G_{11}^{-1} \quad (9)$$

In Eq. 9 $\mathrm{BRG}_{c_{11}}$ is the block relative gain given in Manousiouthakis et al. (1980), that is,

$$\mathrm{BRG}_{c_{11}} = [\mathrm{I} - G_{cd_{11}}G_{11}^{-1}\; G_{cr_{11}}G_{c_{11}}^{-1}]^{-1} \quad (10)$$

Note that Eq. 9 gives the first column of S, Eq. 5. The other $k - 1$ columns of S are also computed from Eq. 9 after substituting G_{ii}, $i = 2, 3, \ldots, k$, for G_{11} and permuting G. In the special case of 2×2 block subsystems ($k = 2$) Eq. 9 simplifies and gives the two relative sensitivities $S_{12} = -(H_1 - I)\,[I + (\mathrm{BRG}_{11}^{-1} - I)\,H_1]^{-1}G_{12}G_{22}^{-1}$ and $S_{21} = -(H_2 - I)\,[I + (\mathrm{BRG}_{22}^{-1} - I)\,H_2]^{-1}G_{21}G_{11}^{-1}$.

The following section illustrates that the particular choice of normalization of responses adopted in the definition of the relative sensitivity, Eq. 4, results in useful properties that can guide the control structure selection process and the design of the decentralized controllers.

Properties of Relative Sensitivities; Implications for Analysis and Design

When a set point change is made in the ith subsystem its effect on the other subsystems is represented by $Y_{c_{ii}i}$. These closed-loop dynamic interactions can be expressed in terms of the relative sensitivities after rearranging Eq. 4, that is, $Y_{c_{ii}i} = S_{c_{ii}i}\cdot Y_{ii}$ $i = 1, 2, \ldots, k$. The magnitude of the interactions, denoted by the maximum singular value σ^*, should be kept as small as possible by imposing.

$$\sigma^*\,(S_{c_{ii}i}\,Y_{ii}) < \delta_i(\omega)\;\forall\omega \text{ and } i = 1, \ldots, k \quad (11)$$

Therefore $\sigma^*\,[S_{c_{ii}i}(\omega)]$ should be small ($\ll 1$) within the same bandwidth for which $\sigma^*(Y_{ii}) \approx 1$. The larger this bandwidth is, the better the dynamic interactions are rejected. Then the obvious question is: what limits the bandwidth? Close examination of the frequency behavior of $S_{c_{ii}i}$ answers this. First consider $i = 1$:

At $\underline{\omega = 0}$, $\quad H_i = I$ and Eq. 9 gives

$$S_{c_{11}1}\,(\underline{\omega = 0}) = 0 \rightarrow S_{j1} = 0\;\forall\, j = 2, 3, \ldots, k \quad (12)$$

Now let us examine the high-frequency behavior. Noting that

$$H_i\,(\omega \rightarrow \infty) = 0, \quad (13)$$

Eq. 9 gives the high-frequency asymptote:

$$S_{c_{11}1}\,(\underline{\omega \rightarrow \infty}) \triangleq \bar{S}_{c_{11}1}(\omega) = G_{cd_{11}}G_{11}^{-1} = \begin{bmatrix} G_{21}G_{11}^{-1} \\ G_{31}G_{11}^{-1} \\ G_{k1}G_{11}^{-1} \end{bmatrix} \triangleq \begin{bmatrix} \bar{S}_{21} \\ \bar{S}_{31} \\ \bar{S}_{k1} \end{bmatrix} \quad (14)$$

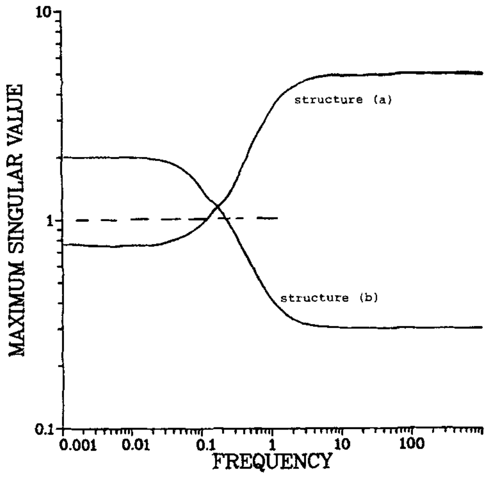

Figure 2. Asymptotes.

The other high-frequency asymptotes $\bar{S}_{c_{ii}}$'s for $i = 2, 3, \ldots, k$ can be similarly derived from $S_{c_{ii}}$ for $i = 2, \ldots, k$ and collected under a single matrix $\bar{S}$ called the relative sensitivity asymptote matrix:

$$\bar{S}(\omega) \triangleq \begin{bmatrix} I & \bar{S}_{12} & \cdot & \bar{S}_{1k} \\ \bar{S}_{21} & I & \cdot\cdot & \\ \cdot & & & \\ \bar{S}_{k1} & \bar{S}_{k2} & & I \end{bmatrix} = \begin{bmatrix} I & G_{12}G_{22}^{-1} & \cdots & G_{1k}G_{kk}^{-1} \\ G_{21}G_{11}^{-1} & I & & G_{2k}G_{kk}^{-1} \\ G_{k1}G_{11}^{-1} & \cdots & & I \end{bmatrix} \quad (15)$$

Then the following important conclusions can be made:

1. *Pairing Rule.* Under closed-loop control $\sigma^* [S_{c_{ii}}(\omega)]$ will start from zero at $\omega = 0$ and eventually reach its open-loop asymptote $\sigma^* [G_{cd_{ii}}(\omega)\ G_{ii}^{-1}(\omega)]$ for $i = 1, 2, \ldots, k$. Consequently, asymptotes that persistently exceed 1 are not desirable since they will decrease the bandwidth within which $\sigma^* [S_{c_{ii}}(\omega)]$, and therefore interactions, can be made small. Since the values of these asymptotes do not depend on the controller tuning, they should be used to select control structures. Specifically, input-output pairings should be chosen so as to minimize the magnitudes of the asymptotes, $\sigma^* [\bar{S}_{c_{ii}}(\omega)]$ for $i = 1, 2, \ldots, k$.

2. *Design Rule.* While relative sensitivity asymptotes are used for pairing, actual relative sensitivities need to be used for controller design. In particular $\sigma^* (S_{c_{ii}})$'s should be made small within the bandwidth of $\sigma^* (Y_{ii})$'s for $i = 1, 2, \ldots, k$. This is achieved by tuning H_i's subject to closed-loop stability (for stability check we recommend the dynamic BRG [Arkun, 1987]).

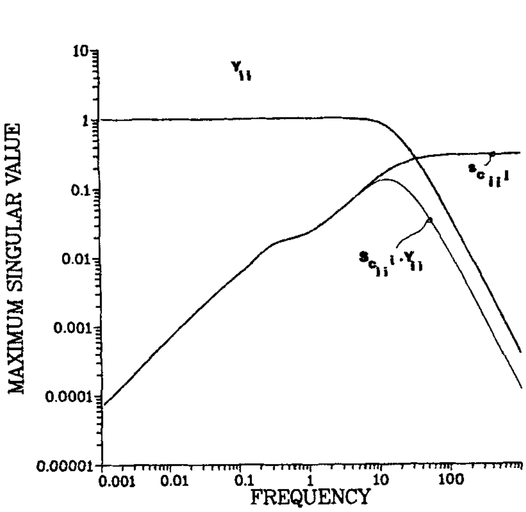

Figure 3. $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.05$.

Example

Consider the 3×3 system studied in Gagnepain and Seborg (1982)

$$G(s) = \begin{bmatrix} \dfrac{-2}{10s+1} & \dfrac{1.5}{5s+1} & \dfrac{1}{s+1} \\ \dfrac{1.5}{5s+1} & -\dfrac{1}{s+1} & \dfrac{2}{10s+1} \\ \dfrac{1}{s+1} & \dfrac{2}{10s+1} & \dfrac{1.5}{5s+1} \end{bmatrix}$$

G is partitioned into three subsystems ($k = 3$) and two alternative SISO decentralized structures are considered:

a. (y_1, u_1), (y_2, u_3), (y_3, u_2)
b. (y_1, u_3), (y_2, u_2), (y_3, u_1)

In the following results $H_i = 1/(\epsilon_i s + 1)^2$ for $i = 1, 2, 3$ where ϵ_i is the tuning parameter. For each alternative structure (a) and (b), the three asymptotes $\sigma^* (\bar{S}_{c_{ii}})$ turn out to be equal, as shown in Figure 2. For structure (a) all three asymptotes persistently exceed 1 at high frequencies, suggesting that dynamic interactions are equally bad in all directions. Therefore (a) is rejected. On the other hand, for (b) all the asymptotes are much smaller than 1 at higher frequencies. Consequently they will not limit the bandwidth of the relative sensitivities. This implies that for (b) interactions are equally small in all directions. This is confirmed by Figure 3 for a representative design $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.05$. Note that the relative sensitivities approach their asymptotes at high frequencies, and because of good asymptotes it was possible to choose a large bandwidth for Y_{ii} and at the same time satisfy $\sigma^* (S_{c_{ii}} \cdot Y_i) \ll 1\ \forall\omega$ and $i = 1, 2, 3$.

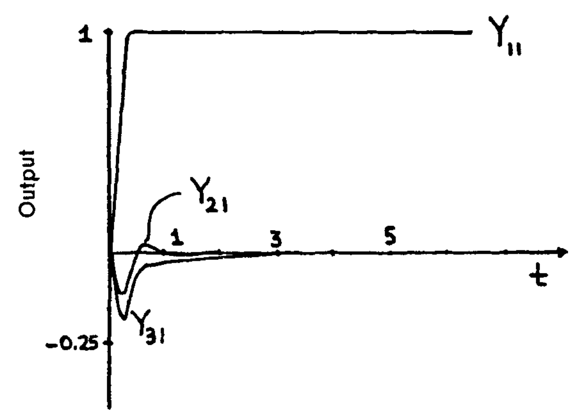

Figure 4. Set point responses.

Some simulation results are shown in Figure 4. The rest of the set point responses show the same trend. For structure (b), because of good relative sensitivities tuning was quite easy while ϵ_i's could be monotonically decreased without encountering any stability and performance problems due to interactions. This was not the case at all for structure (a). Finally, we should note that for structure (b) the lowest value of ϵ and thus the final design will be determined by the plant uncertainty and physical constraints on inputs, which are not addressed here.

Acknowledgment

Financial support from the National Science Foundation through the Grant No. CBT-85/32/3 is gratefully acknowledged. We also acknowledge the help of Georgios Charos and Sean McClenaghan with the calculations.

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Manuscript received Oct. 1, 1987, and revision received Nov. 2, 1987.